Problem 9.23 If you take the model in Ex. 4.1 at face value, what natural frequency do you get? Put in the actual numbers. Where, in the electromagnetic spectrum, does this lie, assuming the radius of the atom is 0.5 Å? ~~Find the coefficients of refraction and dispersion and compare them with those for hydrogen at 0°C and atmospheric pressure: $A = 1.36 \times 10^{-4}$, $B = 7.7 \times 10^{-15}\ \mathrm{m}^2$.~~

Problem 9.24 Find the width of the anomalous dispersion region for the case of a single resonance at frequency ω_0. Assume $\gamma \ll \omega_0$. Show that the index of refraction assumes its maximum and minimum values at points where the absorption coefficient is at half-maximum.

Problem 9.27 Show that the mode TE_{00} cannot occur in a rectangular wave guide. [*Hint:* In this case $\omega/c = k$, so Eqs. 9.180 are indeterminate, and you must go back to 9.179. Show that B_z is a constant, and hence—applying Faraday's law in integral form to a cross section—that $B_z = 0$, so this would be a TEM mode.]

Problem 9.28 Consider a rectangular wave guide with dimensions 2.28 cm $\times$ 1.01 cm. What TE modes will propagate in this wave guide, if the driving frequency is 1.70×10^{10} Hz? Suppose you wanted to excite only *one* TE mode; what range of frequencies could you use? What are the corresponding wavelengths (in open space)?

Problem 9.30 Work out the theory of TM modes for a rectangular wave guide. In particular, find the longitudinal electric field, the cutoff frequencies, and the wave and group velocities. Find the ratio of the lowest TM cutoff frequency to the lowest TE cutoff frequency, for a given wave guide. [*Caution*: What is the lowest TM mode?]

Homework #3 Solutions

Thursday, April 23, 2009
3:21 PM

9.23

$$E = -\frac{1}{4\pi\varepsilon_0}\cdot\frac{qx}{a^3} \qquad F = qE = -kx$$

where $k = -\frac{qE}{x} = \frac{q^2}{4\pi\varepsilon_0 a^3}$

$$\omega_0 = \sqrt{\frac{k}{m}} = \frac{q}{\sqrt{4\pi\varepsilon_0 a^3}}$$

$$\nu = \frac{\omega_0}{2\pi} = 7.14\cdot10^{15}\,Hz$$

$$\lambda = \frac{c}{\nu} = \frac{3\cdot10^8}{7.1\cdot10^{15}} \approx 4\cdot10^{-8}\,m \quad \text{OR} \quad 40\,nm$$

Ultraviolet!

9.24

$$n = 1 + A\frac{x}{x^2+\gamma^2\omega^2} \approx 1 + A\frac{x}{x^2+\gamma^2\omega_0^2}$$

where $x = \omega_0^2 - \omega^2$

$$\frac{\partial n}{\partial x} = \frac{A}{x^2+\gamma^2\omega_0^2} - \frac{2x^2A}{(x^2+\gamma^2\omega_0^2)^2} = 0$$

$$\frac{\gamma^2\omega^2 - x^2}{(x^2+\gamma^2\omega^2)^2} = 0$$

$x = \pm\gamma\omega_0$

(we assumed $\omega \approx \omega_0$ or that $\left|\frac{x}{\omega_0^2}\right| = \frac{\gamma}{\omega_0} << 1$) True!

$$\alpha = \frac{A}{x^2+\gamma^2\omega^2} \approx \frac{A}{x^2+\gamma^2\omega_0^2} \quad (\omega\approx\omega_0)$$

$$\alpha_{max} = \frac{A}{\gamma^2\omega_0^2} \quad \text{when } x=0$$

$$\alpha_{1,2}(x=\pm\gamma\omega_0) = \frac{A}{(\pm\gamma\omega_0)^2+\gamma^2\omega_0^2} = \frac{A}{2\gamma^2\omega_0^2}$$

OR $\alpha_{1,2} = \frac{\alpha_{max}}{2}$

9.27 TE_{00} $E_z = 0$

$k = \omega/c$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = i\omega B_z$$

$$-ikE_y = i\omega B_x$$

$$ikE_x = i\omega B_y$$

$$\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = 0$$

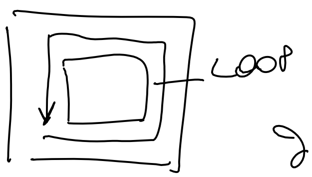
$$\frac{\partial B_z}{\partial y} - ikB_y = -\frac{i\omega}{c^2}E_x$$

$$ikB_x - \frac{\partial B_z}{\partial x} = -\frac{i\omega}{c^2}E_y$$

$$\frac{\partial B_z}{\partial y} = ikB_y - \frac{i\omega}{c^2}\cdot\frac{\omega B_y}{k} = ikB_y - ikB_y = 0$$

$$\frac{\partial B_z}{\partial x} = -ik\cdot\frac{k}{\omega}E_y + \frac{i\omega}{c^2}E_y = 0$$

Since $\frac{\partial B_z}{\partial y} = \frac{\partial B_z}{\partial x} = 0$ for all x and y, $B_z(x,y) = \text{const.}$



$$\oint E \cdot dl = -A \cdot \int \frac{\partial B_z}{\partial t}$$

$$\frac{\partial B_z}{\partial t} = i\omega\, B_z = \text{const}$$

Since $E=0$ inside conductor,

$$\oint E \cdot dl = 0 \Rightarrow B_z = 0$$

9.28 $a = 2.28\,\text{cm}$

$b = 1.01\,\text{cm}$

$$\nu_{mn} = \frac{\omega_{mn}}{2\pi} = \frac{c}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

$$\nu_{01} = \frac{c}{2b} = 1.4 \cdot 10^{10}\,\text{Hz} \quad < 1.7 \cdot 10^{10}\,\text{Hz} = \nu$$

$$\nu_{10} = \frac{c}{2a} = 0.66 \cdot 10^{10}\,\text{Hz} < \nu$$

$$\nu_{11} = \frac{c}{2}\sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{b}\right)^2} = 1.62 \cdot 10^{10}\,\text{Hz} < \nu$$

$$\nu_{20} = 2 \cdot \nu_{10} = 1.32 \cdot 10^{10}\,\text{Hz} < \nu$$

All others $(\nu_{21}, \nu_{12}, \nu_{30}, \nu_{02}, \text{etc.})$ are $> \nu$

9.30 $E_z(x,y) = X(x)\,Y(y)$

$$X(x) = A \cdot \sin k_x x + B \cos k_x \cdot x$$

$E_z = 0$ at $x = 0$, $x = a$

$B = 0$, and

$$k_x = \frac{\pi m}{a}$$

Similarly for $Y(y)$:

$$E_z = E_0 \cdot \sin\frac{m\pi x}{a} \cdot \sin\frac{n\pi y}{b}$$

$$\frac{\omega_{mn}}{c} = \sqrt{\left(\frac{\pi m}{a}\right)^2 + \left(\frac{\pi n}{b}\right)^2}$$

Lowest TM mode is (11)

$$\omega_{11}^{TM} = c\pi\sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{b}\right)^2}$$

Lowest TE mode is (10)

$$\omega_{10}^{TE} = \frac{c\pi}{a}$$

$$\frac{\omega_{11}^{TM}}{\omega_{10}^{TE}} = \sqrt{1 + \left(\frac{a}{b}\right)^2}$$